Property of Some Common Fixed Points in Complex Valued Metric Space

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Abstract

In this paper, we purpose some properties of two mappings to prove some common fixed point theorems of $S,T:X\rightarrow X$ satisfying (CLR_S) and (CLR_T) properties and some property in complex valued metric space (X, d). This work extends and improves some results of Ali (Ali, 2016).

Keywords: Complex valued metric space, Common fixed point, Weakly compatible, CLR property, Property E.A.

Introduction

The axiomatic development of a metric space was essentially carried out by French mathematician Frechet in the year 1906 (Frechet, 1906). [After the Banach contraction principle, because of various applications. Many mathematics used the Banach contractive principle to study an existence and uniqueness of fixed points]. Banach fixed point theorem in a complete metric space introduced by Banach (Banach, 1922) because it was generalized inmany spaces. In 2011, Azam, Fisher and Kham (Azam et al., 2011), introduced the notion of complex valued metric spaces and established sufficient condition for the existence of common fixed points of a pair of mappings and satisfying contractive condition. Complex valued metric space is a generalization of classical metric space. Bhatt, Chaukiyal and Dimri (Bhatt et al., 2011) have proved a theorem of common fixed point for weakly compatible mapping in complex valued metric space. Recently, Ali (Ali, 2016) obtains some common fixed point theorems for a pair of weakly compatible mapping satisfying (CLR) property in complex valued metric space.

Theorem 1.1 (Ali, 2016) Let (X, d) be a complex valued metric space and $S,T:X\rightarrow X$ be weakly compatible mappings such that

(i) S and T satisfy (CLR_S) property and,

(ii) $d(Tx,Ty)\precsim \lambda d(Sx,Sy)+\dfrac{\mu d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}, \forall\, x,y\in X,$

where λ,μ are nonnegative real numbers with $\lambda+\mu<1$. Then S and T have a unique common fixed point.

The aim of this paper is to introduce and obtain some conditions in **Theorem 1.1**, then we prove the uniqueness of S and T with weakly compatible mappings of satisfying (CLR_S) and (CLR_T) property, respectively.

Preliminaries

Let $\boldsymbol{C}$ be the set of complex numbers and $z_1, z_2 \in \boldsymbol{C}$. Define a partial order relation "$\precsim$" on $\boldsymbol{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \operatorname{Re}(z_1) \le \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \le \operatorname{Im}(z_2).$$

Thus $z_1 \precsim z_2$ if one of the followings holds:

(1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

(2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

(3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

(4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) are satisfied and we will write $z_1 \prec z_2$ only (4) is satisfied.

Remark 2.1 (Sintunavarat and Kumam, 2011) We can easily check the following:

(i) If $a, b \in \boldsymbol{R}$, $0 \le a \le b$ and $z_1 \precsim z_2$ then $az_1 \precsim bz_2$, $\forall z_1, z_2 \in \boldsymbol{C}$.

(ii) $0 \precsim z \precnsim z_2 \Rightarrow |z_1| < |z_2|$.

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Azam et al. (Azam, A., Brain, F., & Khan, M., 2011) defined the complex valued metric space in the following way:

Definition 2.2 (Azam et al., 2011) Let X be a nonempty set. Suppose that the mapping $d: X \times X \longrightarrow \boldsymbol{C}$ satisfies the following conditions:

(C1) $0 \precsim d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(C2) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(C3) $d(x,z) \precsim d(x,y) + d(y,z)$, for all $x, y, z \in X$.

Then d is called a complex valued metric and (X,d) is called a complex valued metric space.

Example 2.3 (Ali, 2016) Let $\boldsymbol{X} = \boldsymbol{C}$. Define the mapping $d: X \times X \longrightarrow \boldsymbol{C}$ by

$$d(z_1, z_2) = i\left|z_1 - z_2\right|, \quad \forall z_1, z_2 \in X.$$

One can easily check that (X,d) is a complex valued metric space.

Definition 2.4 (Azam et al., 2011) Let (X,d) be a complex valued metric space. Then

(i) A sequence $\{x_n\}$ in X converge to $x \in X$ if for every $0 \prec r \in \boldsymbol{C}$ there exists $m \in \boldsymbol{N}$ such that $d(x_n, x) \prec r$ for all $n > m$. We denote this by $\lim_{n\to\infty} x_n = x$

(ii) If for every $0 \prec r \in \boldsymbol{C}$ there exists $m \in \boldsymbol{N}$ such that $d(x_n, x_{n+k}) \prec r$ for all $k \in \boldsymbol{N}$ then $\{x_n\}$ is called a Cauchy sequence in (X,d).

(iii) If every Cauchy sequence in X is convergent in X then (X,d) is called a complete complex valued metric space.

Definition 2.5 (Azam et al., 2011) Let (X,d) be completed complex valued metric space, then

(i) S and T are said to be commuting if

$$STx = TSx, \quad \text{for all } x \in X$$

(ii) S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

(iii) S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$, that is they commute at their coincidence points.

From definition 2.5 we knew that the commuting is weakly compatible.

Definition 2.6 (Verma and Pathak, 2013) Let (X,d) be a complex valued metric space. The self-maps S and T are said to satisfy the property *(E.A)* if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t.$$

for some $t \in X$.

The (CLR_g) property is more powerful than property *(E.A),* which was defined by Sintunavarat and Kumam (Sintunavarat and Kumam, 2011), in a metric space for a pair of self-mappings.

Definition 2.7 (Sintunavarat and Kumam, 2011) Suppose that (X,d) is a complex valued metric space and $f, g : X \longrightarrow X$. Then f and g are said to satisfy the (CLR_g) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

Example 2.8 (Ali, 2016) Let X=$\boldsymbol{C}$ and d be any complex valued metric on X. Define $f, g : X \longrightarrow X$ by $fz = 2z + i$ and $gz = 3z - 1$ for all $z \in X$. Consider a sequence $\{z_n\} = \{i + 1 + \frac{1}{n}\}$. Then

$$\lim_{n\to\infty} fz_n = \lim_{n\to\infty}[2(i+1+\frac{1}{n})+i] = 3i+2, \text{ and}$$

$$\lim_{n\to\infty} gz_n = \lim_{n\to\infty}[3(i+1+\frac{1}{n})-1] = 3i+2 = g(i+1),$$

We see that $x = i+1$. Thus f and g satisfy the (CLR_g) property.

Lemma 2.9 (Azam et al., 2011) Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to $x\in X$ if and only if $|d(x_n,x)| \to 0$ as $n\to\infty$.

Lemma 2.10 (Azam et al., 2011) Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n,x_{n+m}) \to 0$ as $n\to\infty$ for every natural number *m.*

Lemma 2.11 (Datta and Ali, 2012) Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then for any $a\in X$, $\lim_{n\to\infty} d(x_n,a) = d(x,a)$.

Results

In this section we give some conditions for a complex valued metric space and prove an unique common fixed point of *S* and *T* with satisfying (CLR_S) and (CLR_T), respectively.

Theorem 3.1 Let (*X, d*) be a complex valued metric space and $S,T: X \to X$ be weakly compatible mappings such that

(i) *S* and *T* satisfy (CLR_S) property and,

(ii) $d(Tx,Ty) \precsim \lambda d(Sx,Sy) + \dfrac{\mu d(Tx,Sy)d(Ty,Sx) + \gamma d(Tx,Sx)d(Sy,Ty)}{1+d(Sx,Sy)}, \forall x,y\in X,$

where λ,μ,γ are nonnegative real numbers with $\lambda+\mu<1$. Then *S* and *T* have a unique common fixed point.

Proof First, we must show that *S* and *T* have a common fixed point. Since *S* and *T* satisfy (CLR_S) property, there exists a sequence $\{x_n\}$ and u in *X* such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Su. \tag{3.1}$$

From condition *(ii),* we have

$$d(Tx_n,Tu) \precsim \lambda d(Sx_n,Su) + \frac{\mu d(Tx_n,Su)d(Tu,Sx_n) + \gamma d(Tx_n,Sx_n)d(Su,Tu)}{1+d(Sx_n,Su)}. \tag{3.2}$$

From (3.2) and by remark 2.1 *(ii),* we have

$$\left|d(Tx_n,Tu)\right| \leq \left|\lambda d(Sx_n,Su) + \frac{\mu d(Tx_n,Su)d(Tu,Sx_n)+\gamma d(Tx_n,Sx_n)d(Su,Tu)}{1+d(Sx_n,Su)}\right|$$

$$\leq \left|\lambda d(Sx_n,Su)\right| + \left|\frac{\mu d(Tx_n,Su)d(Tu,Sx_n)+\gamma d(Tx_n,Sx_n)d(Su,Tu)}{1+d(Sx_n,Su)}\right|$$

$$= |\lambda|\left|d(Sx_n,Su)\right| + \frac{\left|\mu d(Tx_n,Su)d(Tu,Sx_n)+\gamma d(Tx_n,Sx_n)d(Su,Tu)\right|}{\left|1+d(Sx_n,Su)\right|}$$

$$\leq \lambda\left|d(Sx_n,Su)\right| + \frac{\mu\,|d(Tx_n,Su)|\,|d(Tu,Sx_n)| + \gamma\,|d(Tx_n,Sx_n)|\,|d(Su,Tu)|}{|1+d(Sx_n,Su)|}. \tag{3.3}$$

Hence

$$\left|d(Tx_n,Tu)\right| \leq \lambda\left|d(Sx_n,Su)\right| + \frac{\mu\,|d(Tx_n,Su)|\,|d(Tu,Sx_n)| + \gamma\,|d(Tx_n,Sx_n)|\,|d(Su,Tu)|}{|1+d(Sx_n,Su)|}. \tag{3.4}$$

From (3.1), we obtain that

$$\lim_{n\to\infty} d(Tx_n,Sx_n) = d(Su,Su) = 0. \tag{3.5}$$

Now from Lemma 2.9 and (3.1) we have

$$\lim_{n\to\infty}\left|d(Tx_n,Su)\right| = 0 \text{ and } \lim_{n\to\infty}\left|d(Sx_n,Su)\right| = 0.$$

From (3.4), it follows that

$$\lim_{n\to\infty}\left|d(Tx_n,Tu)\right| = 0.$$

And then $\lim_{n\to\infty} Tx_n = Tu$, we obtain that $Su = Tu$. Since S and T are weakly compatible, we have

$$TTu = TSu = STu = SSu. \tag{3.6}$$

Again from condition *(ii),* we have

$$d(Tx_n,TSu) \precsim \lambda d(Sx_n,SSu) + \frac{\mu d(Tx_n,SSu)d(TSu,Sx_n)+\gamma d(Tx_n,Sx_n)d(SSu,TSu)}{1+d(Sx_n,SSu)}. \tag{3.7}$$

From (3.7) and $SSu = TSu$, we have

$$d(Tx_n,TSu) \precsim \lambda d(Sx_n,SSu) + \frac{\mu d(Tx_n,SSu)d(SSu,Sx_n)+\gamma d(Tx_n,Sx_n)(0)}{1+d(Sx_n,SSu)}$$

$$= \lambda d(Sx_n,SSu) + \frac{\mu d(Tx_n,SSu)d(SSu,Sx_n)}{1+d(Sx_n,SSu)}. \tag{3.8}$$

Thus

$$d(Tx_n,TSu) \precsim \lambda d(Sx_n,SSu) + \frac{\mu d(Tx_n,SSu)d(SSu,Sx_n)}{1+d(Sx_n,SSu)}. \tag{3.9}$$

Since $\lim_{n\to\infty} Sx_n = Su$ and by Lemma 2.11, we obtain that

$$\lim_{n\to\infty} d(Sx_n,TSu) = d(Su,TSu). \tag{3.10}$$

From $SSu = TSu$ and (3.10), we have

$$\lim_{n\to\infty} d(Sx_n, SSu) = d(Su, TSu). \tag{3.11}$$

From $\lim_{n\to\infty} Tx_n = Su$ and Lemma 2.11, it follows that

$$\lim_{n\to\infty} d(Tx_n, TSu) = d(Su, TSu) = d(Su, SSu) = \lim_{n\to\infty} d(Tx_n, SSu).$$

From (3.9), Lemma 2.11 and letting $n \longrightarrow \infty$, we have

$$\begin{aligned} d(Su, SSu) &\precsim \lambda d(Su, SSu) + \frac{\mu d(Su, SSu) d(Su, SSu)}{1 + d(Su, SSu)} \\ &\precsim (\lambda + \mu) d(Su, SSu). \end{aligned}$$

From remark 2.1 (ii), it follows that

$$\begin{aligned} \left| d(Su, SSu) \right| &\le \left| (\lambda + \mu) d(Su, SSu) \right| \\ &= | \lambda + \mu || d(Su, SSu) | \\ &= (\lambda + \mu) | d(Su, SSu) |. \end{aligned}$$

Hence $$[1 - (\lambda + \mu)] \, | d(Su, SSu) | \le 0. \tag{3.12}$$

Since $0 \le \lambda + \mu < 1$, we have $| d(Su, SSu) | = 0$ and hence $SSu = Su$. Therefore $TSu = SSu = Su$, it follows that Su is a common fixed point of S and T.

Finally, we prove the uniqueness part of the result, suppose that $Sw = Tw = w$ for some $w \in X$. Since $Tu = Su$, it follows that $d(Tu, Su) = 0$ and condition *(ii),* we have

$$\begin{aligned} d(Tu, w) = d(Tu, Tw) &\precsim \lambda d(Su, Sw) + \frac{\mu d(Tu, Sw) d(Tw, Su) + \gamma d(Tu, Su) d(Sw, Tw)}{1 + d(Su, Sw)} \\ &\precsim \lambda d(Su, Sw) + \frac{\mu d(Tu, Sw) d(Tw, Su)}{1 + d(Su, Sw)}. \end{aligned} \tag{3.13}$$

Since,

$$\mu d(Su, w) d(w, Su) \precsim \frac{\mu d(Su, w)(1 + d(Sw, Su))}{1 + d(Su, w)} = \mu d(Su, w), \tag{3.14}$$

replace (3.14) into (3.13), it follows that

$$\begin{aligned} d(Su, w) = d(Tu, w) &\precsim \lambda d(Su, w) + \mu d(Su, w) \\ &= (\lambda + \mu) d(Su, w). \end{aligned}$$

By remark 2.1 (ii), we obtain that

$$\begin{aligned} \left| d(Su, w) \right| &\le \left| (\lambda + \mu) d(Su, w) \right| \\ &= (\lambda + \mu) \left| d(Su, w) \right|. \end{aligned}$$

Since $0 \le \lambda + \mu < 1$, we have $\left| d(Su, w) \right| = 0$ and so $Su = w$. Therefore, Su is a unique common fixed point of S and T. □

Corollary 3.2 Let (X, d) be a complex valued metric space and $S, T : X \rightarrow X$ be a commuting such that

(i) S and T satisfy (CLR_S) property and,

(ii) $d(Tx,Ty) \precsim \lambda d(Sx,Sy) + \dfrac{\mu d(Tx,Sy)d(Ty,Sx) + \gamma d(Tx,Sx)d(Sy,Ty)}{1+d(Sx,Sy)}, \forall x,y \in X,$

where λ, μ, γ are nonnegative real numbers with $\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof From Definition 2.5, we have the commuting implied weakly compatible. Then S and T are weakly compatible. By Theorem 3.1 we have Su is the unique common fixed point of S and T. ☐

Corollary 3.3 (Ali, 2016) Let (X, d) be a complex valued metric space and $S,T: X \rightarrow X$ be a weakly compatible mappings such that

(i) S and T satisfy (CLR_S) property and,

(ii) $d(Tx,Ty) \precsim \lambda d(Sx,Sy) + \dfrac{\mu d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)}, \forall x,y \in X,$

where λ, μ are nonnegative real with $\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof From condition (ii) in Theorem 3.1, we have

$$\begin{aligned} d(Tx,Ty) &\precsim \lambda d(Sx,Sy) + \frac{\mu d(Tx,Sy)d(Ty,Sx)}{1+d(Sx,Sy)} \\ &\precsim \lambda d(Sx,Sy) + \frac{\mu d(Tx,Sy)d(Ty,Sx) + \gamma d(Tx,Sx)d(Sy,Ty)}{1+d(Sx,Sy)} \end{aligned}$$

for all $x, y \in X$. Then we have the results of Ali (Ali S., 2016). ☐

Theorem 3.4 Let (X, d) be a complex valued metric space and $S,T: X \rightarrow X$ be a weakly compatible mappings such that

(i) S and T satisfy (CLR_T) property,

(ii) $TX \subset SX$ and,

(iii) $d(Tx,Ty) \precsim \lambda d(Sx,Sy) + \dfrac{\mu d(Tx,Sy)d(Ty,Sx) + \gamma d(Tx,Sx)d(Sy,Ty)}{1+d(Sx,Sy)}, \forall x,y \in X,$

where λ, μ, γ are nonnegative real with $\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof From S and T satisfy (CLR_T) property, there exists a sequence $\{x_n\}$ and v in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Tv. \qquad (3.15)$$

Since $TX \subset SX$ and $Tv \in TX$, there exists $u \in X$ such that $Tv = Su$. Thus S and T satisfy (CLR_S) property. Hence, similarly in Theorem 3.1, we have Su is the unique common fixed point of S and T, this complete the proof. ☐

Theorem 3.5 Let (X, d) be a complex valued metric space and $S,T: X \rightarrow X$ be a weakly compatible mappings such that

(i) S and T satisfy property *(E.A.)*

(ii) SX is a complete subspace of X and

$$\text{(iii)}\ d(Tx,Ty) \precsim \lambda d(Sx,Sy) + \frac{\mu d(Tx,Sy)d(Ty,Sx) + \gamma d(Tx,Sx)d(Sy,Ty)}{1 + d(Sx,Sy)}, \forall x,y \in X,$$

where λ, μ, γ are nonnegative real with $\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof From S and T satisfy property *(E.A.)*, there exists a sequence $\{x_n\}$ and t in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t. \quad (3.16)$$

Since SX is a complete subspace of X, there exists $u \in X$ such that $Su = t$. Thus S and T satisfy (CLR_S) property. Hence, similarly in Theorem 3.1, we get that Su is the unique common fixed point of S and T, this complete the proof. □

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