ระบบบิวิลี่ในทีพโฟเรซ์แบบบริโดยิงใน
การแก้ปัญหาการแข่งตัว

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บทคัดย่อ

ระบบบิวิลี่ในทีพโฟเรซ์แบบบริโดยิงเป็นระบบบิวิลี่ที่มีประโยชน์ในการต้านวางแผนล่าและ
ของปัญหาการแข่งตัว เนื่องจากมีความสามารถในการควบคุมและระดมเรือในการต้านทาน อย่างไรก็ตามดิสทรี
โดยทั่วไประบบบิวิลี่ที่มีอยู่ในเรือทั้งนี้จะให้ได้ระดับภูมิต้านทานในการต้านทานภูมิล่าและส่านแหล่ง
ของข้อมูลเฉพาะช่วงเรือเล็กและเรือกลาง ความมีประสิทธิ์ของการต้านวางแผนล่าและแสดงให้เห็นถึงความสามารถ
การควบคุมระดมเรือในการปล่อยสตันตราของแหล่งที่ข้อมูลของข้อมูลระหว่างของเรือเล็กและเรือกลาง และของเรือ
แม้แต่ ในการวิจัยนี้ระบบนี้วิธีการได้ใช้ข้อมูลที่ถูกต้องปรับปรุงเพื่อให้ความมีประสิทธิ์ในการควบคุมและ
ได้รับการใช้ เพื่อแก้ปัญหาการแข่งตัวที่มีรีได้ที่จะรับผลมา ถือวิธีการเปรียบเทียบตัวเพื่อเปรียบ
ประสิทธิภาพของระบบบิวิลี่ที่นำมา

คำสำคัญ: ระบบบิวิลี่ในทีพโฟเรซ์เรือ, บริโดยิง, การแข่งตัว, การเปลี่ยนสอง

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A Fixed-Grid Finite-Difference Method for Solidification Problems

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Abstract

For a numerical solution of solidification problems, a fixed-grid finite-difference method is a popular approach due to the ease of its implementation and less computational effort than the moving-grid system. However, the simple discretisation of the fixed-grid method has a tendency to oscillate numerically in temperature and solid-liquid interface position. The numerical oscillations arise from improper handling of the evolution of the latent heat at the solid-liquid interface. In this work, a modified discretisation is described such that the numerical error could be overcome. A one-dimensional solidification problem with known analytical solution is applied to demonstrate the efficiency of the method.

Keywords: finite difference method, fixed grid, solidification, phase change

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Introduction

Solidification is a heat transfer controlled process by which a liquid changes to solid. Thermal effects during solidification are of great interest in many engineering applications such as in the making of ice, the freezing of food, thermal energy storage, growth of crystals, casting and welding of metals, etc. The effect of solidification is most evident when casting is the final operation. Since a solidification structure and its associated defects are difficult to eliminate once they are created, good control of the solidification at the starting point is therefore most important [Flemings, M. C. 1974].

Solidification is governed by nonlinear equations and the complexity is that the solid-liquid interface is moving as the latent heat is released at the interface; as a result, the location of the interface is a priori unknown and must follow as a part of solution. The numerical technique is difficult due to the presence of this highly nonlinear moving boundary condition. To overcome this numerical difficulty, a great many methods have been developed [Muelibauer, J. C. and Sunderland, J. 1965., Voller, V. R., Cross, M. and Markatos, N. C. 1987., Schneider, G. E. 1987. and Lee, S. L., Tzong, R. Y. 1991.]. However, most of the finite-difference methods predicted oscillatory temperature and interface due to improper handling of the evolution of the latent heat at the interface. The oscillation would give an incorrect solution and prevent the numerical solution from converging.

The objective of this work is to develop a fixed-grid finite-difference method which removes numerical oscillation in temperature and solid-liquid interface position. An interfacial energy-balance equation was used to track the solid-liquid interface in combination with a modified discretisation of heat fluxes crossing control surfaces. A one-dimensional solidification problem which known analytical solution is applied as a numerical example.

Mathematical Model

Solidification due to temperature change at one end of pure material in an enclosure (Figure 1) has been studied; the wall at $x' = 0$ and $x' = L$ are isothermal while the others are adiabatic. Initially, the cavity is filled with an isothermal liquid ($T_{i} > T_{m}$). At $t' > 0$, the temperature is lowered below the melting temperature ($T_{i} < T_{m}$) at the left wall, and solidification is initiated thereafter. The interface propagates into the liquid and the thickness of the solid increases with the passage of time.
The energy equation is expressed in terms of temperature, with a source term (in square bracket) representing an amount of latent heat liberated at the solid-liquid interface:

\[
\rho' c'_p \frac{\partial T'}{\partial t'} = \frac{\partial}{\partial x'} \left( k' \frac{\partial T'}{\partial x'} \right) - \left[ \rho' h_j \frac{\partial f_j}{\partial t'} \right]
\]  

(1)

where \( \rho, c_p, k, t \) represent density, specific heat, thermal conductivity and time. \( h_j \) is the latent heat and \( f_j \) is the liquid fraction which is unity in the liquid phase and zero in the solid phase. Thus, \( f_j \) falls from unity to zero after a liquid is completely solidified. Primes denote dimensional quantities.

The solid-liquid interface is tracked with the use of an interfacial energy equation:

\[
\rho' h_j \frac{\partial x_j'}{\partial t'} = \left( k'_s \frac{\partial T'}{\partial x'} - k'_l \frac{\partial T_j'}{\partial x'} \right)
\]  

(2)

where the solid-liquid interface is given by \( x' = x'_j(t') \). \( s \) and \( l \) denote solid and liquid, respectively. After introducing a non-dimensional transformation, defined by

\[
T = \frac{(T' - T_0)}{\Delta T'}; \quad \rho = \frac{\rho'}{\rho'_0}; \quad c_p = \frac{c'_p}{c'_{p,e}}; \quad x = \frac{x'}{L};
 Ste = \frac{c'_{p,e}(T'_m - T_0)}{h_j}; \quad t = t'k'_s/(\rho'c'_pL^2); \quad k = k'/k'_j; \quad x_j = \frac{x'_j}{L}.
\]  

(3)

\( \Delta T' \) can be either \( T'_m - T_0' \) or \( T'_l - T_0' \). Assuming constant properties, equations (1) and (2) become

\[
\rho' c'_p \frac{\partial T'}{\partial t'} = \frac{\partial}{\partial x'} \left( k' \frac{\partial T'}{\partial x'} \right) - \left[ \frac{1}{\text{Ste}} \frac{\partial f_j}{\partial t'} \right],
\]  

(4)

\[
\frac{\partial x_j'}{\partial t'} = \text{Ste} \left( k'_s \frac{\partial T_j'}{\partial x'} - k'_l \frac{\partial T_j'}{\partial x'} \right).
\]  

(5)

**One-Dimensional Finite-difference Discretisation**

The energy equation is discretised by a direct substitution of backward difference approximations in time and central difference approximations in space, together with an application of semi-implicit scheme:

\[
\frac{T^*_i - T^{i-1}_i}{\Delta t} = \left( k_{i-\frac{1}{2}} \frac{1}{(\Delta x)^2} \left[ \frac{1}{2}(T^*_i + T^{i-1}_i) - \frac{1}{2}(T^{i-1}_i + T^{i-2}_i) \right] - k_{i+\frac{1}{2}} \frac{1}{(\Delta x)^2} \left[ \frac{1}{2}(T^{i+1}_i + T^{i-1}_i) - \frac{1}{2}(T^{i}_i + T^{i-1}_i) \right] \right) \frac{1}{\text{Ste}} \frac{f_{i-1} - f_{i}}{\Delta t},
\]  

(6)

where \( i \) and \( k \) are notation indexes for a control volume and time, respectively. \( k_{i-\frac{1}{2}} \) and \( k_{i+\frac{1}{2}} \) are thermal conductivities at boundaries of the control volume \( i \).
In solidification problems, temperature gradient may be discontinuous across the solid-liquid interface due to the latent heat liberated at the interface. The discontinuous temperature gradient is controlled by a Stefan number and a liquid-solid thermal conductivity ratio as shown in Figure 2. The discontinuous temperature gradient becomes large with decreasing Stefan number or increasing values of $k_i / k_s$.

![Figure 2](image_url) Temperature profiles

In a simple discretisation, a piecewise-linear profile of temperature (Figure 3a) is assumed between mesh points [Patankar, S. V. 1980.]. This simple profile assumption cannot handle the discontinuity and produces some spatial oscillations. A linearised profile (Figure 3b) is proposed in order to represent the interface by a surface rather than a control volume. To account for the discontinuity of temperature gradient, thermal conductivities at boundaries of the solidifying control-volume are calculated by equation (7).
The interface is tracked with the use of the interfacial energy-balance equation, discretised as

$$\frac{x_i^k - x_{i-1}^k}{\Delta t} = \text{Ste} \left( k_i \frac{T_i - T_{i-1}}{x_i - x_{i-1}} - k_{i-1} \frac{T_{i-1} - T_{i-2}}{x_{i-1} - x_{i-2}} \right)$$  

(8)

Once the interface position is found, the liquid fraction is calculated by

$$f_{i,j} = \begin{cases} 
1 & \text{for } x_i < x_{i-\frac{1}{2}} \\
\frac{x_{i+\frac{1}{2}} - x_i}{\Delta x} & \text{for } x_{i-\frac{1}{2}} < x_i < x_{i+\frac{1}{2}} \\
0 & \text{for } x_i > x_{i+\frac{1}{2}}
\end{cases}$$  

(9)

**Numerical Solutions**

The finite-difference equations obtained in the previous section are solved following the procedures described below.

**Step 1.** Specify the initial and boundary conditions. Specify time step (\(\Delta t\)) such that the initial movement of the interface is less than a half of grid size (\(x_i < 0.5 \Delta x\)). Initially using a small time step could overcome difficulties caused by a sudden change in boundary condition in the initial stage.

**Step 2.** Compute thermal conductivities at control-volume boundaries by equation (7).

**Step 3.** Compute interface location and liquid fraction by equation (8) and (9).

**Step 4.** Compute temperature from equation (6) using the tri-diagonal matrix algorithm.

**Step 5.** Repeat steps 2 through 4 until the temperature values are converged. Convergence criterion was \(\varepsilon_T = 10^{-6}\), where \(\varepsilon_T = \frac{1}{\text{NX}(\max|T_i^n - T_i^{n-1}|)} \sum_{i=1}^{\text{NX}} |T_i^n - T_i^{n-1}| \).

Here

**Step 6.** For the problem, the given by

alpha is thermal capacity

For Tzong[6]: solid were the
Here, the subscript $n$ is the iteration index and $NX$ is the number of nodes.

**Step 6.** For the next time step, repeat steps 2 through 5 with the data at the previous time step.

### Numerical Example

One-dimensional solidification (Stefan problems) in a half space ($x \geq 0$) was used as a test problem. The analytical solution of the Stefan problem is known as Neumann's solution [2] and is given by

$$x_i(t) = 2\lambda\sqrt{\alpha t}$$  \hspace{1cm} (10)

$$T(x,t) = \begin{cases} 
\frac{T_m - T_0}{\text{erf}(\Lambda)} \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) + T_0 & ; x < x_i \\
T_m & ; x = x_i \\
T_i - \left( \frac{T_i - T_m}{\text{erfc}(\Lambda,\sqrt{\alpha / \alpha_i})} \right) \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) & ; x > x_i 
\end{cases}$$ \hspace{1cm} (11)

where $\Lambda$ is obtained from

$$\frac{e^{-\Lambda^2}}{\text{erf}(\Lambda)} + \frac{k_i}{k_s} \sqrt{\frac{\alpha_i}{\alpha}} \left( \frac{T_m - T_i}{T_m - T_0} \right) \frac{e^{-\Lambda^2(\alpha_i/\alpha)}}{\text{erfc}(\Lambda,\sqrt{\alpha_i / \alpha_i})} = \frac{\lambda \sqrt{\pi}}{\text{Ste}}.$$ \hspace{1cm} (12)

$\alpha$ is thermal diffusivity, defined as $\alpha = k / (\rho c_p)$.

For comparison purposes, the parameters were assigned the same values as Lee and Tzong[6]: $\text{Ste} = 1.2$ and $[T_i, T_m, T_0] = [1.667, 1, 0]$. The properties for both the liquid and the solid were the same.
Figure 4 shows the interface position from the present method and analytical solution for solidification period up to \( t = 0.015 \). The numerical results by Voller et al. [Voller, V. R., Cross, M., and Markatos, N. C. 1987.], Schneider [Schneider, G. E. 1987.], and Lee and Tzong [Lee, S. L., Tzong, R. Y. 1991.] were also plotted for comparisons. All numerical solutions used the same grid spacing (\( \Delta x = 0.02 \)) and time step (\( \Delta t = 0.0001 \)). The interface positions produced by Voller and Schneider are seen to produce zigzag variations. Voller and Schneider have simulated the enthalpy change at a grid point as the latent heat released from the corresponding control volume. The control volume will suddenly release its entire latent heat at the instant while the solid-liquid interface is sweeping through its grid point. Hence, the interface positions were approximated from the ratio of the volumetric latent heat and total enthalpy. This would result in a zigzag function with jumps having the size of grid spacing.

The present solution and Lee and Tzong's solution are smooth and close to the analytical solution. This is because the latent heat is represented by the liquid fraction \( f \) for each control volume such that the latent heat can be released evenly. However, the relative error of 2% is observed from Lee and Tzong's solution since the interface position was calculated by using the simplified interfacial energy-balance equation in which the transient term was neglected. The transient term was included in the present method and hence the predicted solution is very close to the analytical solution with a relative error of order 0.2%. Good agreements with the analytical solution were also observed for calculation up to \( t = 0.2 \) by interface position (Figure 4b), temperature history at \( x = 0.1 \) (Figure 5a) and temperature profiles (Figure 5b).

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Flemings, N. Muehlbauer, M. Putankar, S. Schneider, C. Voller, V. R. P.
A Fixed-Grid Finite-Difference Method for Solidification Problems

This paper has described an efficient fixed-grid finite-difference method for solidification problems. The solid-liquid interface was tracked with the use of the interfacial energy-balance equation. The modified discretization was obtained in terms of the thermal conductivities at the control volume boundaries, to account for the discontinuity of temperature gradient in the vicinity of the solid-liquid interface. A one-dimensional half-space solidification problem was computed to validate the present method. This strategy has been proved to have an excellent performance as it can remove numerical oscillation when compared with other numerical solutions of the fixed-grid method. The present method can also be extended to two-dimensional problems without much difficulty for directional solidification whereas the analytical solution is applicable only to a simpler one-dimensional problem.

Conclusion

References