Solutions of the Diophantine Equation \( x^2 + py^2 = z^2 \)

Piyanut Puangjumpa*
Department of Mathematics and Statistics, Faculty of Science and Technology, Surindra Rajabhat University, 32000 Thailand
*Corresponding Author: piyanut@math.sut.ac.th

Received: 12 January 2016; Revised: 6 December 2016; Accepted: 13 December 2016; Available online: 1 April 2017

Abstract

This paper is to identify the Diophantine equation \( x^2 + py^2 = z^2 \) where \( p \) is a prime number and \( x, y \) and \( z \) are integers satisfies; case 1: \( p = 2 \), has no integer solution if \( y \) is odd, and have integer solutions \((x, y, z)\) is \((\pm(2\alpha - \beta), \pm\sqrt{\alpha\beta}, \pm(2\alpha + \beta))\) where \( \alpha\beta \) is a square number if \( y \) is even, case 2: \( p \neq 2 \), have integer solutions \((x, y, z)\) is \((\pm\frac{p\alpha - \beta}{2}, \pm\sqrt{\alpha\beta}, \pm\frac{p\alpha + \beta}{2})\) where \( \alpha\beta \) is an odd square number if \( y \) is odd, and \((\pm(p\alpha - \beta), \pm\sqrt{\alpha\beta}, \pm(p\alpha + \beta))\) where \( \alpha\beta \) is a square number if \( y \) is even.

Keywords: Diophantine equation; Congruence; Integer solutions

1. Introduction

In 1995, Wiles [1] showed that the Diophantine equation \( x^n + y^n = z^n \), \( x, y, z \in \mathbb{Z} \) has not had integer solutions when \( n \geq 3 \). In 1999, Bruin [2] studied the Diophantine equation \( x^2 + y^2 = z^2 \) and \( x^2 + y^4 = z^5 \). In 2004, Bennett [3] found the solution of the Diophantine equation \( x^2 + y^2 = z^2 \). In 2014, Abdelalim and Dyani [4] searched the solution of the Diophantine equation \( x^2 + y^2 = z^2 \). In 2015, Abdelalim and Diany [5] characterized the solutions of the Diophantine equation \( x^2 + y^2 = z^2 \).

Thus, this paper aims to study the Diophantine equation \( x^2 + py^2 = z^2 \) where \( p \) is a prime number and \( x, y \) and \( z \) are integers.
2. Materials and Methods

Lemma 1: The Diophantine equation \( x^2 + 2y^2 = z^2 \) has no integer solution where \( x, y \) and \( z \) are integers with \( y \) is odd.

Proof:

Suppose that \((x, y, z)\) is a solution of the Diophantine equation \( x^2 + 2y^2 = z^2 \) with \( y \) is odd.

Since \( y^2 \equiv 1 \pmod{4} \) then \( 2y^2 \equiv 2 \pmod{4} \), these consider into 2 cases as follow:

Case 1: If \( x \) is odd then \( z \) is odd.
Thus, \( x^2 \equiv 1 \pmod{4} \) then \( x^2 + 2y^2 \equiv 3 \pmod{4} \). This is a contradiction with \( z^2 \equiv 1 \pmod{4} \).

Case 2: If \( x \) is even then \( z \) is even.
Thus, \( x^2 \equiv 0 \pmod{4} \) then \( x^2 + 2y^2 \equiv 2 \pmod{4} \). This is a contradiction with \( z^2 \equiv 0 \pmod{4} \).

Therefore, by case 1 and 2, the Diophantine equation \( x^2 + 2y^2 = z^2 \) has no integer solution where \( x, y \) and \( z \) are integers with \( y \) is odd.

Lemma 2: The Diophantine equation \( x^2 + 2y^2 = z^2 \) has the solutions in the form

\[
(x, y, z) = \left( \pm (2\alpha - \beta), \pm \alpha \sqrt{2\beta}, \pm (2\alpha + \beta) \right)
\]

where \( x, y \) and \( z \) are integers with \( y \) is even, \( \alpha \beta \) is a square number.

Proof:

Let \((x, y, z)\) be a solution of the Diophantine equation \( x^2 + 2y^2 = z^2 \) with \( y \) is even.

So, we have

\[
x^2 + 2y^2 = z^2
\]

\[
2y^2 = z^2 - x^2
\]

\[
2y^2 = (|z| - |x|)(|z| + |x|).
\]

So that, \( 2(|z| - |x|) \) hence, \( 2(|z| - |x|) \) or \( 2(|z| + |x|) \), we have \( x, z \) are even or \( x, z \) are odd, then \( |z| - |x| \) and \( |z| + |x| \) are even, we have \( \frac{|z| - |x|}{2}, \frac{|z| + |x|}{2} \) are integers. It follows that,

\[
\frac{2y^2}{4} = \frac{|z| - |x|}{4} \frac{|z| + |x|}{4}
\]

\[
2 \left( \frac{y^2}{2} \right) = \left( \frac{|z| - |x|}{2} \right) \left( \frac{|z| + |x|}{2} \right).
\]

Hence, \( \frac{|z| - |x|}{2} \) then \( \frac{|z| + |x|}{2} \), these consider into 2 cases as follow:
Case 1: If \( \left| \frac{z - |x|}{2} \right| \) so that, there exists an integer \( \alpha \) such that \( 2\alpha = \left| \frac{z - |x|}{2} \right| \) and let \( \beta = \left| \frac{z + |x|}{2} \right| \). It follows that,

\[
2\left( \frac{y}{2} \right)^2 = (2\alpha)\beta \\
\left( \frac{y}{2} \right)^2 = \alpha\beta
\]

\[
\frac{y}{2} = \pm \sqrt{\alpha\beta} \text{, where } \alpha\beta \text{ is a square number;}
\]

Since \( 2\alpha = \left| \frac{z - |x|}{2} \right| \) and \( \beta = \left| \frac{z + |x|}{2} \right| \) then \( |x| = -(2\alpha - \beta) \) and \( |z| = 2\alpha + \beta \) that is, \( x = \pm(2\alpha - \beta) \) and \( z = \pm(2\alpha + \beta) \).

Case 2: If \( \left| \frac{z + |x|}{2} \right| \) so that, there exists an integer \( \alpha \) such that \( 2\alpha = \left| \frac{z + |x|}{2} \right| \) and let \( \beta = \left| \frac{z - |x|}{2} \right| \). It follows that,

\[
2\left( \frac{y}{2} \right)^2 = \beta (2\alpha) \\
\left( \frac{y}{2} \right)^2 = \alpha\beta
\]

\[
\frac{y}{2} = \pm \sqrt{\alpha\beta} \text{, where } \alpha\beta \text{ is a square number;}
\]

Since \( 2\alpha = \left| \frac{z + |x|}{2} \right| \) and \( \beta = \left| \frac{z - |x|}{2} \right| \) then \( |x| = 2\alpha - \beta \) and \( |z| = 2\alpha + \beta \).

That are \( x = \pm(2\alpha - \beta) \) and \( z = \pm(2\alpha + \beta) \). Therefore, by case 1 and 2, the solutions of the Diophantine equation are

\[
(x, y, z) = \left( \pm(2\alpha - \beta), \pm2\sqrt{\alpha\beta}, \pm(2\alpha + \beta) \right)
\]

where \( \alpha\beta \) is a square number.
3. Results and Discussion

Theorem 3: The Diophantine equation \( x^2 + py^2 = z^2 \) has the solutions in the form

\[
(x, y, z) = \left( \pm \frac{p\alpha - \beta}{2}, \pm \sqrt{\alpha\beta}, \pm \frac{p\alpha + \beta}{2} \right)
\]

where \( p \) is an odd prime number, \( x, y, z \) are integers with \( y \) is odd and \( \alpha\beta \) is an odd square number.

Proof:

Let \((x, y, z)\) be a solution of the Diophantine equation \( x^2 + py^2 = z^2 \) with \( y \) is odd. We have,

\[
x^2 + py^2 = z^2
\]

\[
py^2 = z^2 - x^2
\]

\[
py^2 = (|z| - |x|)(|z| + |x|).
\]

Hence, \( p\mathbb{I}(|z| - |x|) \) then \( p\mathbb{I}(|z| + |x|) \), these consider into 2 cases as follow:

Case 1: If \( p\mathbb{I}(|z| - |x|) \) so that, there exists an integer \( \alpha \) such that \( p\alpha \equiv |z| - |x| \) and let \( \beta = |z| + |x| \). It follows that,

\[
py^2 = (p\alpha)\beta
\]

\[
y^2 = \alpha\beta
\]

\[
y = \pm\sqrt{\alpha\beta}, \text{ where } \alpha\beta \text{ is an odd square number.}
\]

Since \( p\alpha \equiv |z| - |x| \) and \( \beta \equiv |z| + |x| \) then \( |x| \equiv -\frac{p\alpha - \beta}{2} \) and \( |z| \equiv \frac{p\alpha + \beta}{2} \).

That are \( x = \pm\frac{p\alpha - \beta}{2} \) and \( z = \pm\frac{p\alpha + \beta}{2} \).

Case 2: If \( p\mathbb{I}(|z| + |x|) \) so that, there exists an integer \( \alpha \) such that \( p\alpha \equiv |z| + |x| \) and let \( \beta = |z| - |x| \). It follows that,

\[
py^2 = \beta(p\alpha)
\]

\[
y^2 = \alpha\beta
\]

\[
y = \pm\sqrt{\alpha\beta}, \text{ where } \alpha\beta \text{ is an odd square number.}
\]

Since \( p\alpha \equiv |z| + |x| \) and \( \beta \equiv |z| - |x| \) then \( |x| \equiv \frac{p\alpha - \beta}{2} \) and \( |z| \equiv \frac{p\alpha + \beta}{2} \).

That are \( x = \pm\frac{p\alpha - \beta}{2} \) and \( z = \pm\frac{p\alpha + \beta}{2} \).

Therefore, by case 1 and 2, the solutions of this equation are in the form

\[
(x, y, z) = \left( \pm \frac{p\alpha - \beta}{2}, \pm \sqrt{\alpha\beta}, \pm \frac{p\alpha + \beta}{2} \right)
\]

where \( \alpha\beta \) is an odd square number.
Theorem 4: The Diophantine equation \( x^2 + py^2 = z^2 \) has the solutions in the form

\[
(x, y, z) = \left( \pm(p\alpha - \beta), \pm 2\sqrt{\alpha\beta}, \pm(p\alpha + \beta) \right)
\]

where \( p \) is a prime number, \( x, y, z \) are integers with \( y \) is even and \( \alpha\beta \) is a square number.

Proof:

Let \((x, y, z)\) be a solution of the Diophantine equation \( x^2 + py^2 = z^2 \) with \( y \) is even.

We have,

\[
x^2 + py^2 = z^2
\]

\[
py^2 = z^2 - x^2
\]

\[
py^2 = (|z| - |x| |z| + |x|)
\]

Since \( y \) is even then \( y^2 \) is divisible by 4, so, \( 4|(|z| - |x| |z| + |x|) \) hence, \( |z| - |x| \) and \( |z| + |x| \) are even, we have

\[
py^2 = \frac{(|z| - |x| |z| + |x|)}{2}
\]

\[
\rho \left( \frac{y}{2} \right)^2 = \frac{|z| - |x|}{2} \quad \text{or} \quad \rho \left( \frac{y}{2} \right)^2 = \frac{|z| + |x|}{2}
\]

Hence, \( \rho \left| \frac{z - |x|}{2} \right| \) then \( \rho \left| \frac{z + |x|}{2} \right| \), these consider into 2 cases as follow:

Case 1: If \( \rho \left| \frac{z - |x|}{2} \right| \) so that, there exists an integer \( \alpha \) such that

\[
\rho\alpha = \frac{|z| - |x|}{2}
\]

and let

\[
\beta = \frac{|z| + |x|}{2}
\]

It follows that,

\[
\rho \left( \frac{y}{2} \right)^2 = (p\alpha) \beta
\]

\[
\left( \frac{y}{2} \right)^2 = \alpha \beta
\]

\[
y = \pm 2\sqrt{\alpha\beta}, \quad \text{where} \quad \alpha \beta \quad \text{is a square number};
\]

\[
y = \pm 2\sqrt{\alpha\beta}
\]

Since \( \rho\alpha = \frac{|z| - |x|}{2} \) and \( \beta = \frac{|z| + |x|}{2} \) then \( |x| = -(\rho\alpha - \beta) \) and

\[
|z| = \rho\alpha + \beta.
\]

That are \( x = \pm(\rho\alpha - \beta) \) and \( z = \pm(\rho\alpha + \beta) \).

Case 2: If \( \rho \left| \frac{z + |x|}{2} \right| \) setting \( \rho\alpha = \frac{|z| + |x|}{2} \) and \( \beta = \frac{|z| - |x|}{2} \). It follows that,
\[
\rho \left( \frac{y}{2} \right)^2 = \beta (\rho \alpha) \\
\left( \frac{y}{2} \right)^2 = \alpha \beta \\
\frac{y}{2} = \pm \sqrt{\alpha \beta}, \text{ where } \alpha \beta \text{ is a square number;}
\]
\[
y = \pm 2 \sqrt{\alpha \beta}
\]
Since \( \rho \alpha = \frac{|z| + |x|}{2} \) and \( \beta = \frac{|z| - |x|}{2} \) then \( |x| \equiv \rho \alpha - \beta \) and \( |z| \equiv \rho \alpha + \beta \).
That are \( x = \pm (\rho \alpha - \beta) \) and \( z = \pm (\rho \alpha + \beta) \).

Therefore, by case 1 and 2, the Diophantine equation has the solutions in the form
\[
(x, y, z) = \left( \pm (\rho \alpha - \beta), \pm \sqrt{\alpha \beta}, \pm (\rho \alpha + \beta) \right)
\]

where \( \alpha \beta \) is a square number.

4. Conclusion

It can be seen that this paper shown the solution of the Diophantine equation \( x^2 + p y^2 = z^2 \) has no integer solution if \( p \) is an even prime number, \( y \) is odd and have integer solutions
\[
(x, y, z) = \left( \pm \rho \alpha - \beta, \pm \sqrt{\alpha \beta}, \pm \rho \alpha + \beta \right)
\]
where \( \alpha \beta \) is an odd square number, \( p \) is an odd prime number
and \( x, y \) and \( z \) are integers with \( y \) is odd. Also, we have seen that the solution of the Diophantine equation \( x^2 + py^2 = z^2 \) is
\[
(x, y, z) = \left( \pm (\rho \alpha - \beta), \pm \sqrt{\alpha \beta}, \pm (\rho \alpha + \beta) \right)
\]
where \( \alpha \beta \) is a square number, \( p \) is a prime number and \( x, y \) and \( z \) are integers with \( y \) is even.

In the future, we may extend this work by extending the value of \( p \) is a prime number to the value of \( n \) is a natural number.

5. Acknowledgement

I highly appreciate the editors and referees for his helpful comments and suggestions to make this study are more valuable. This paper was supported by the Faculty of Science and Technology, Surindra Rajabhat University.

6. References

[2] N. Bruin, The Diophantine equation \( x^2 \pm y^4 = \pm z^6 \) and \( x^2 + y^8 = z^3 \), COMPOS MATH. 118 (1999) 305 – 321.
