Irrational Numbers with Generalized Bounded Partial Quotients

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Abstract

The definition of irrational numbers with generalized bounded partial quotients is defined. We will investigate some relation between irrational numbers with bounded partial quotients in simple continued fraction and in generalized continued fraction. Furthermore, using three distance theorem of Slater, we also find the result of three distance theorem for generalized continued fraction in some kind of sequence.

Keywords: continued fraction; bounded partial quotients; three distance theorem.

1. Introduction

A continued fraction is an expression of the form

\[ \alpha_0 + \frac{\beta_1}{\alpha_1 + \frac{\beta_2}{\alpha_2 + \cdots}} \]

where \( \alpha_0 \) and \( \beta_n \) may be any kind of numbers, variables, or functions. The number of term can be either finite or infinite. If \( \alpha_0 \in \mathbb{Z}, \alpha_n \in \mathbb{N} \) and \( \beta_n = 1 \) for all \( n \geq 1 \), then the expression is called a simple continued fraction; otherwise, it is called a generalized continued fraction. For simplicity, we denote the term

\[ \frac{\beta_1}{\alpha_1 + \frac{\beta_2}{\alpha_2 + \cdots}} \]

by \( \frac{\beta_n}{\alpha_n + \cdots} \).

It is also known that all irrational numbers can be uniquely expressed as a simple continued fraction. Moreover, simple continued fractions expansion of any real number is finite if and only if that number is rational.

For examples,

\[ \frac{19}{8} = 2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2}}} \]

and

\[ \frac{13}{95} = \frac{1}{7 + \frac{1}{3 + \frac{1}{4}}} \]

Thus, we can determine that a fixed irrational number \( \theta \) has infinite simple continued fraction expansion:

\[ \theta = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \cdots}}} \]

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\[ \theta = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}, \]

where \( a_0 \in \mathbb{Z} \) and \( a_n \in \mathbb{N} \) for all \( n \geq 1 \). We start with introducing some definition about simple continued fraction.

**Definition 1.1** We say that \( \theta \) has bounded partial quotients if and only if \( \sup_{n \geq 1} a_n < \infty \).

**Definition 1.2** Define integers \( p_n \) and \( q_n \) by: \( p_0 = 1, p_0 = a_0, p_n = a_n p_{n-1} + p_{n-2} \) for \( n \geq 1 \), \( q_0 = 0, q_1 = 1, q_n = a_n q_{n-1} + q_{n-2} \) for \( n \geq 1 \) and define \( \eta_n = \left| q_n \theta - p_n \right| \) for all \( n \geq -1 \).

Remark: For all \( n \geq 1 \), \( \frac{p_n}{q_n} \) is called \( n \) th convergent to \( \theta \) and satisfy \( \frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}; \)
in fact, \( \lim_{n \to \infty} \frac{p_n}{q_n} = \theta \).

**Definition 1.3** For any real number \( x \), let \( \lfloor x \rfloor \) denote the largest integer less than or equal to \( x \) and let \( \{x\} \) denote the fractional part of \( x \), i.e., \( \{x\} = x - \lfloor x \rfloor \).

For \( N \in \mathbb{N} \), consider the sequence of distinct points \( \{1\theta, 2\theta, \ldots, N\theta\} \) in \([0,1]\), arranged in increasing order: \( 0 < \lfloor k_1 \theta \rfloor < \lfloor k_2 \theta \rfloor < \cdots < \lfloor k_n \theta \rfloor < 1 \), where \( k_j \in \{1, 2, \ldots, N \} \) for \( j = 1, 2, \ldots, N \). Then, interval \([0,1]\) is partitioned into \( N + 1 \) subintervals. A set of these subintervals are a partition of \([0,1]\). Let \( P_\theta (N) \) denote an aforementioned partition. The maximum and minimum lengths of the subintervals of \( P_\theta (N) \) are denoted, respectively, by \( d_\theta (N) \) and \( d' \theta (N) \).

In 1998, Mukherjee and Kaner proved that \( \theta \) has bounded partial quotients if and only if the sequence \( \left( \frac{d_\theta (N)}{d' \theta (N)} \right)_{N=1}^{\infty} \) is bounded by using a theorem proved by Slater (1967) as follows:

**Theorem 1.4** (three distance theorem)

(I) For all \( N \in \mathbb{N} \), \( N \) can be uniquely represented as \( N = r q_k + q_{k-1} + s \), for uniquely integer \( k \geq 0 \), \( 1 \leq r \leq a_{k+1} \), and \( 0 \leq s < q_k \).

(II) For the partition \( P_\theta (N) \), represent \( N \) as \( N = r q_k + q_{k-1} + s \) in part (I), then there are \((r-1)q_k + q_{k-1} + s + 1\) subintervals of length \( \eta_k \), \( s + 1 \) subintervals of length \( \eta_{k-1} - \eta_k \), and \( q_k - (s + 1) \) subintervals of length \( \eta_{k-1} - (r-1) \eta_k \).

Because \( \theta \) can be expressed as an infinite generalized continued fraction:

\[ \theta = a_0 + \frac{\beta_1}{a_1 + \frac{\beta_2}{a_2 + \ddots}}, \]

the definition of irrational numbers with bounded partial quotients may be established. In this paper, we attend in case that \( a_0 \) is an integer, \( a_n \) and \( \beta_n \) are positive integer for all \( n \geq 1 \). However, for a given sequence \( \left( \frac{\beta_n}{a_n} \right)_{n=1}^{\infty} \) of positive integer, \( \theta \) may have more than one generalized continued fraction expansion. For example, let \( \theta = \sqrt{2} \), \( \beta_n = 3 \) for all \( n \geq 1 \), we get

\[ \sqrt{2} = 1 + \frac{3}{2 + \frac{3}{6 + \frac{3}{2 + \ddots}}}, \quad \sqrt{2} = 1 + \frac{3}{7 + \frac{3}{12 + \frac{3}{2 + \ddots}}}. \]
\[ \sqrt{2} = 1 + \frac{3}{1} + \frac{3}{8} + \frac{3}{3} + \ldots. \]

Hence, we must obviate this ambiguousness for aforementioned purpose.

2. Numbers with generalized bounded partial quotients

In this section, we provide the definition of numbers with bounded partial quotients in generalized continued fraction form and investigate some basic property related with numbers of simple continued fraction form.

First of all, we start with proving two lemmas.

**Lemma 2.1** For any sequences \((a_n)_{n=0}^{\infty}, (\beta_n)_{n=1}^{\infty}\) of complex number, where \(a_n, \beta_n \neq 0\) for all \(n \geq 1\). If we define complex numbers \(p_n\) and \(q_n\) by:

- \(p_{n-1} = 1, p_0 = a_0, p_n = a_0 p_{n-1} + \beta_n p_{n-2}\) for \(n \geq 1\), \(q_{n-1} = 0, q_0 = 1, q_n = a_n q_{n-1} + \beta_n q_{n-2}\) for \(n \geq 1\), then \(p_n = a_0 + K_{j=m}^{n} \beta_j q_n\) and \(q_n p_{n-1} - q_{n-1} p_n = (-1)^n \beta_1 \beta_2 \cdots \beta_n\) for all \(n \geq 1\).

**Proof.** For any sequences \((a_n)_{n=0}^{\infty}, (\beta_n)_{n=1}^{\infty}\) of complex number, we see that:

\[ \frac{p_n}{q_n} = a_0 + \frac{\beta_n}{a_1} \]

and \(q_n p_n - q_{n-1} p_{n-1} = -\beta_1\). By mathematical induction, assume that \(p_n = a_0 + \sum_{j=m}^{n} \beta_j q_n\) and \(q_n p_{n-1} - q_{n-1} p_n = (-1)^n \beta_1 \beta_2 \cdots \beta_n\), we obtain that \(q_{n+1} p_n - q_n p_{n+1} = -\beta_1 q_{n+1} p_{n-1} - q_{n-1} p_{n+1} = (-1)^{n+1} \beta_1 \beta_2 \cdots \beta_{n+1}\).

Since \(\frac{a_{n+1} + \beta_{n+1} p_{n+1}}{a_{n+1} q_{n+1} + \beta_{n+1} q_{n+1}} = \frac{p_n}{q_n} = a_0 + \sum_{j=m}^{n} \beta_j q_n\) and \(q_n p_{n-1} - q_{n-1} p_n = (-1)^n \beta_1 \beta_2 \cdots \beta_n\), we replace the term \(a_n\) by \(a_n + \frac{\beta_{n+1}}{a_{n+1}}\) we obtain

\[ \alpha_n + K_{j=m}^{n+1} \beta_j q_n = \left( \frac{\alpha_n + \beta_{n+1}}{\alpha_{n+1}} \right) p_{n+1} + \beta_n p_{n-2} \]

We conclude that \(p_n = a_0 + K_{j=m}^{n} \beta_j q_n\) and \(q_n p_{n-1} - q_{n-1} p_n = (-1)^n \beta_1 \beta_2 \cdots \beta_n\) for all \(n \geq 1\).

**Lemma 2.2** For a given sequence \((B_n)_{n=1}^{\infty}\) of positive integer, \(\theta\) can be uniquely written as

\[ \theta = A_0 + \sum_{n=1}^{\infty} \frac{B_n}{A_n} = A_0 + \frac{B_1}{A_1} + \frac{B_2}{A_2} + \cdots \]

where \(A_n, B_n \in \mathbb{N}\) and \(0 < B_n / A_n < 1\) for all \(n \geq 1\).

**Proof.** Let \(A_0 = [\theta], \quad \theta = \{\theta\};

\[ A_n = \frac{B_n}{\theta - 1}, \quad \theta = \left\{ \frac{B_n}{\theta - 1} \right\}, \quad \text{for all } n \geq 1. \]

We see that \(A_n\) is an integer, \(A_n\) are positive integer, \(0 < B_n / A_n < 1\) and \(B_n / A_n \leq A_n\) for all \(n \geq 1\). By mathematical induction, we have

\[ \theta = A_0 + \frac{B_1}{A_1} + \frac{B_2}{A_2} + \cdots, \quad \text{for all } n \geq 1. \]

We can define integers \(p_n\) and \(q_n\) as in Lemma 2.1, then we have

\[ A_0 + K_{j=m}^{n} \frac{B_j}{A_j} = \frac{p_n}{q_n} = A_0 p_{n-1} + B_n p_{n-2}. \]

By replacing

\[ A_n = A_0 q_{n-1} + B_n q_{n-2}, \]

by \(\theta = \frac{(A_0 + \theta) p_{n-1} + B_n p_{n-2}}{(A_0 + \theta) q_{n-1} + B_n q_{n-2}}\)

\[ \theta = \frac{p_n + \theta q_n}{q_n + \theta q_{n-1}} \]

get

\[ \left| \frac{\theta - p_n}{q_n} \right| = \frac{| \frac{p_n + \theta q_n - p_n}{q_n + \theta q_{n-1}} |}{| q_n + \theta q_{n-1} |} = \left( q_n + \theta q_{n-1} \right) \left| \frac{\theta (q_n + \theta q_{n-1})}{q_n + \theta q_{n-1}} \right| = \frac{|q_n + \theta q_{n-1}|}{q_n + \theta q_{n-1}}. \]
\[ B_n \equiv B \ldots B \theta_n . \] By mathematical induction, we have \( 0 < B \ldots B \theta_n < 1 \) for all \( n \geq 1 \). Since
\[ \lim_{n \to \infty} \frac{1}{Q_n + \theta_1 Q_{n-1}} = 0 \] and by the squeeze theorem, we have
\[ \lim_{n \to \infty} \left| \theta - \frac{P_n}{Q_n} \right| = 0. \] Thus, \( \theta = \lim_{n \to \infty} \frac{P_n}{Q_n} = \lim_{n \to \infty} \left( A_0 + \sum_{j=1}^{n} B_j \frac{1}{A_j} \right) \), i.e., \( \theta = A_0 + \frac{B_1}{A_1 + \frac{B_2}{A_2 + \cdots}} \).

By mathematical induction, we obtain
\[ 0 < \theta_{n-1} = \frac{\infty}{j=a} B_j < 1 \] for all \( n \geq 1 \). Hence, \( \theta \) can be written as \( \theta = A_0 + \frac{\infty}{j=a} B_j \), where \( 0 < \frac{\infty}{j=a} B_j < 1 \) for all \( n \geq 1 \). For uniqueness, let \( \theta = C_0 + \frac{\infty}{j=a} B_j \), where \( C_0 \in \mathbb{Z} \) and \( C_n \in \mathbb{N} \) and \( 0 < \frac{\infty}{j=a} B_j < 1 \) for all \( n \geq 1 \). Since \( C_0 \) is an integer and \( 0 < \frac{\infty}{j=a} B_j < 1 \), we have \( C_0 = \lfloor \theta \rfloor = A_0 \) and \( \frac{\infty}{j=a} B_j \theta_0 \). By mathematical induction, assume that \( C_n = A_n \) and \( \frac{\infty}{j=a} B_j \theta_s \). Then \( C_{n+1} = \frac{\infty}{j=a} B_j \theta_s \).

Since \( C_{n+1} \) is an integer and \( 0 < \frac{\infty}{j=a} B_j < 1 \), we obtain \( C_{n+1} = \left[ \frac{B_{n+1}}{\theta_n} \right] = A_{n+1} \) and \( \theta = \frac{\infty}{j=a} B_j \).

By Lemma 2.2, let \( B_n \) be a fixed sequence of positive integer, we can write \( \theta \) as a form in the Lemma. So we have \( \theta = a_0 + \frac{\infty}{j=a} B_j \), where \( a_0 \in \mathbb{Z}, a_0 \in \mathbb{Z}, a_n \in \mathbb{N} \), \( A_n \in \mathbb{N} \) and \( 0 < \frac{\infty}{j=a} B_j < 1 \) for all \( n \geq 1 \). Then there is the definition of \( \theta \) convergent to \( \theta \) for generalized continued fraction which as in Lemma 2.1.

**Definition 2.5** Define integers \( p \) and \( q \) by: \( P_{-1} = 1 \), \( P_0 = A_0 \), \( P_n = A_n P_{n-1} + B_n P_{n-2} \) for all \( n \geq 1 \), \( Q_{-1} = 0 \), \( Q_0 = 1 \), \( Q_n = A_n Q_{n-1} + B_n Q_{n-2} \) for all \( n \geq 1 \) and define \( \mu_n = |\theta - p| \) for all \( n \geq -1 \).

Now we can define numbers with bounded partial quotients in generalized continued fraction form and get some preliminary fact as follows:

**Definition 2.3** We say that \( \theta \) has bounded \( (B_n)_{n=1}^{\infty} \)'s partial quotients if and only if \( \sup_{n \geq 1} A_n < \infty \).

**Theorem 2.4** For all \( n \geq 1 \), \( B_n \leq A_n \) and if the sequence \( (B_n)_{n=1}^{\infty} \) is not bounded, then \( \theta \) has not bounded \( (B_n)_{n=1}^{\infty} \)'s partial quotients.

**Proof.** Let \( n \geq 1 \). Since \( \frac{\infty}{j=a} B_j < 1 \) and \( \frac{\infty}{j=a} B_j < 1 \), \( B_n < A_n \) and \( \frac{\infty}{j=a} B_j < 1 \). Since \( A_n \) and \( B_n \) are integer, \( B_n \leq A_n \). Hence, if the sequence \( (B_n)_{n=1}^{\infty} \) is not bounded, then \( (A_n)_{n=1}^{\infty} \) is not bounded i.e. \( \theta \) has not bounded \( (B_n)_{n=1}^{\infty} \)'s partial quotients.
integer such that \( \theta \) also has bounded \((B_n)_{n=1}^{\infty}\)’s partial quotients. Theorem 2.4 gives we a necessary condition. In this paper, we provide a basic example of sufficient condition for \((B_n)_{n=1}^{\infty}\) such that \( \theta \) has bounded \((B_n)_{n=1}^{\infty}\)’s partial quotients. 

From now on we attend in case that the sequence \((B_n)_{n=1}^{\infty}\) satisfying \( B_{2n-1} = B_{2n} \leq a_{2n}\) for all \( n \geq 1 \). Before reach to the next lemma, for simplicity, we define \( f(n) = \frac{1 - (-1)^n}{2} = \begin{cases} 0; n \text{ is even} \\ 1; n \text{ is odd} \end{cases} \) for all integer \( n \).

**Lemma 2.6** For all \( n \geq 0 \),
\[ A_n = \begin{cases} a_n; n \text{ is even} \\ B_n a_n; n \text{ is odd} \end{cases} = (f(n)(B_n - 1) + 1) a_n . \]

**Proof.** For \( n = 0 \), we have \( a_0 = \begin{cases} \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta \theta 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If \( n \) is odd, then \( Q_{n+1} = A_{n+1}Q_n + B_{n+1}Q_{n-1} \). By Lemma 2.6, we have
\[
Q_{n+1} = a_{n+1}Q_n + B_{n+1}Q_{n-1} = B_0B_2\cdots B_{n+1}(a_{n+1}q_n + q_{n-1}) = B_0B_2\cdots B_{n+1}q_{n+1}
\]

Hence, \( Q_{n+1} = \begin{cases} B_0B_2\cdots B_{n+1}q_{n+1} & \text{if } n+1 \text{ is even} \\ B_0B_2\cdots B_{n+1}q_{n+1} \cdot n+1 \text{ is odd} \end{cases} \).

Similarly, we obtain
\[
Q_n = B_0B_2\cdots B_{n+f(a)}q_a \quad \text{for all } n \geq -1.
\]

Finally, we get the result of three distance theorem for generalized continued fraction, which is our main result, for the sequence \( \left( B_n \right)_{n=1}^{\infty} \) which satisfied \( B_{2n-1} = a_{2n} \) for all \( n \geq 1 \).

**Theorem 2.9** (three distance theorem for generalized continued fraction).

(I) For all \( N \in \mathbb{N} \), \( N \) can be uniquely represented as 
\[
N = r \frac{Q_k}{B_0B_2\cdots B_{k+f(k)-1}} + \left( \frac{Q_{k-1}}{B_0B_2\cdots B_{k+f(k)-1}} \right) + s \quad \text{in} \quad B_0B_2\cdots B_{k+f(k)}
\]
part (I), then there are \( (r-1) \mu_k, s+1 \) subintervals of length \( \mu_k \), and
\[
\begin{align*}
&\left( r \frac{Q_k}{B_0B_2\cdots B_{k+f(k)}} \right), & \text{and} & \left( \frac{Q_{k-1}}{B_0B_2\cdots B_{k+f(k)}} \right) \\
&\left( r \frac{\mu_k}{B_0B_2\cdots B_{k+f(k)}} \right) & \text{subintervals of length} & \left( \frac{\mu_k}{B_0B_2\cdots B_{k+f(k)}} \right) \\
&\left( r \frac{1}{B_0B_2\cdots B_{k+f(k)}} \right) & \text{for } k \geq 0, & 1 \leq r \leq \frac{A_{k+1}}{f(k+1)B_{k+1} - 1}, \text{ and} & 0 \leq s < \frac{Q_k}{B_0B_2\cdots B_{k+f(k)}}.
\end{align*}
\]

**Proof.** The proof of this theorem follows straight from Theorem 1.4 and Lemma 2.8.

3. References
